Paper

Comparison between Singular Stress Fields of Sharp V-Notched, Kirchhoff and Mindlin Plates

.

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This study is concerned with the singular stress analysis of sharp V-notched plates under the transverse bending based on the Kirchhoff and the Mindlin plates theories. To apply the fracture mechanics, it is necessary to understand the characteristics of the singular stress fields for the both theories. The similarities and the dissimilarities between the aspects of the both singular stress fields are examined and clarified.

Key Words: *transverse bending, sharp V-notch, Kirchhoff theory, Mindlin theory, fracture mechanics*

1. Introduction

The plate bending problems are normally dealt with the two-dimensional problems for their tractability although their three-dimensional properties. Therefore, the problems involve certain complexity and/or contradictions different from the so-called plane problems. To correct these contradictions, many plate theories[1)] has been published so far. Especially, for the application of fracture mechanics, it is necessary to clarify the characteristics of the singular stress fields between the plate theories.

The Kirchhoff plate theory is the most classical and celebrated theory, and has been adopted extensively for many plate bending problems[2),3),4)]. In the Kirchhoff theory, the transverse shear deformation effects are neglected for thin plate assumption. Although the neglect of shear deformation is negligible for thin plates, this negligence will lead to erroneous quantity for thick plates.

To overcome this shortcoming of Kirchhoff plates, many plate theories taking account of the transverse shear deformations has been published so far. Especially, the Mindin plate theory[5)] takes account of the effect of transverse shear deformation by allowing the normal to undergo constant rotations during deformation. This theory is a simple theory in the many shear deformation theories proposed until now.

In the previous papers[6),7)], the authors have been investigated the method of determination of singular stress fields of sharp V-notched plates by the experimental method based on the Kirchhoff and the Mindlin plate theories, where the similarities and the dissimilarities concerning the results between the two theories are revealed in the behavior of their singular stress fields.

This paper deals with the singular stress fields with respect to the sharp V-notched plates under the transverse bending based on the Kirchhoff and Mindlin Plate theories. The eigenfuntion expansion method is adopted to analyze the singular stress fields, and examined the similarities and dissimilarities between their singular behavior of two theories.

2. Basic Equations of the Kirchhoff plate (KP) and the Mindlin Plate (MP)

2.1 Basic equations

Fig.1 shows the (KP) or the (MP) of thickness h, with a sharp V-notch under free stresses along the notch surface subjected to a uniform transverse bending moment M. The origin of the cylindrical coordinates (r, θ, z) set on the notch tip such that θ=0 axis is the bisector of the notch angle 2β (α+β=π) and coincide with the Cartesian x-axis.

(1) Displacement equations

The displacement equations of the (KP) and the (MP) can be expressed as

(KP): $$Dw_{,\alpha\alpha\beta\beta} = q(x, y) \tag{1}$$

(MP): $$(D/2)\{(1-\nu)\psi_{\alpha,\beta\beta} + (1+\nu)\psi_{\beta,\beta\alpha}\} - \kappa^2 Gh(\psi_\alpha + w_{,\alpha}) = 0 \tag{2}$$

$$\kappa^2 Gh(\psi_{\alpha,\alpha} + w_{,\alpha\alpha}) + q(x, y) = 0 \tag{3}$$

$$(\alpha, \beta = 1,2)$$

where in eq. (1) and eq. (2), w is the displacement in the direction of z, q (x, y) is the distributed load, D is bending rigidity, G is shear modulus, ν is Poisson's ratio, and κ is the correction factor introduced by Mindlin[5].

The displacements in the direction of (r, θ, z) in eqs. (2) and eq. (3) are

$$u_r = z\psi_r(r,\theta),\ u_\theta = z\psi_\theta(r,\theta),\ u_z = w(r,\theta) \tag{4}$$

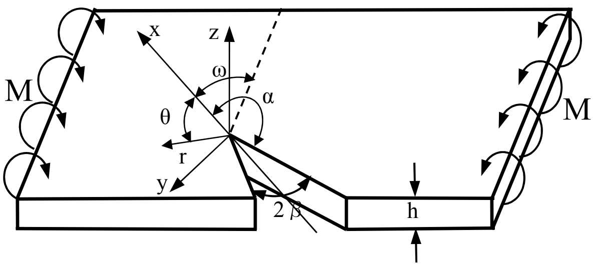

Fig.1 A V- notched plate and coordinate systems system systems

respectively, Representations in the two-dimensional polar coordinates (r, θ) of eq. (1) , eq. (2) and eq. (3) are given by Appendix A.

(2) Boundary conditions

The bounary conditions of the (KP) and (MP) are given by

(KP): $$M_{\theta\theta}(r, \pm\alpha) = 0, Q_\theta(r, \pm\alpha) + \partial M_{r\theta}/\partial r = 0 \tag{5), 6}$$

(MP): $$M_{\theta\theta}(r,\pm\alpha) = 0,\ M_{r\theta}(r,\pm\alpha) = 0,\ Q_\theta(r,\pm\alpha) = 0 \tag{7), (8), (9}$$

where M and Q are the moment and the shearing force, respectively. Eq. (5) and eq. (6) are well known approximate boundary condition of the (KP) theory, and cannot give independently by the shearing force and the twisting moment as the external forces. On the other hand, eq. (7), (8) and eq. (9) in the (MP) theory can be given as the three external forces independently.

3. Similarities and Dissimilarities of Singular Stress Fields of the (KP), and Discussion

In this chapter, representing the analysis process of singular stress fields of the (KP) and the (MP), and clarify the behavior of singular stress character between the both plates, such as characteristic equations, singular stress fields, generalized stress intensity factors (GSIFs) and others.

3.1 Characteristic equations

The characteristic equations of the two plate (KP) and (MP) can be obtained by the eigenfunction expansion method. The analysis process of the (MP) is given by Appendix B, and the case of (KP) is given in the reference[6]. The results are

(KP): $$\sin 2\alpha\lambda - C\lambda \sin 2\alpha = 0 \tag{10}$$

$$\sin 2\alpha\gamma + C\gamma \sin 2\alpha = 0 \tag{11}$$

where $$C = (1-\nu)/(3+\nu)$$

(MP): $$\sin 2\alpha\lambda + \lambda \sin 2\alpha = 0 \tag{12}$$

$$-\sin 2\alpha\gamma + \gamma \sin 2\alpha = 0 \tag{13}$$

$$\sin 2\alpha\delta = 0 \tag{14}$$

where the characteristic values of λ, γ and δ are the order of stress singularities of mode I, mode II and modo III, respectively. Hereafter, since the mode III V-notch analysis was treated in the reference (8), and we do not treat here. From Eq. (10) and eq. (11), the eigenvalues are influenced by the notch angles as well as Poisson's ratio, and are different from the characteristic equations of plane problem[9)]. This is a shortcoming of the (KP) theory, because the surface of plate under uniform bending moment should become the case of two-dimensional problem. On the other hand, eqs. (12) and (13) are coincident with those of the plane problem[9)].

3.2 Definituion of generalized stress intensity factors (GSIFs)

The generalized stress intensity factors (GSIFs) are ordinarily defined as follows:

(KP):

$$K_I = \lim_{r\to 0}\sqrt{2\pi}r^{1-\lambda}\sigma_{\theta\theta}(r,0) \tag{15}$$

$$K_{II} = \lim_{r\to 0}\sqrt{2\pi}Cr^{1-\gamma}\sigma_{r\theta}(r,0) \tag{16}$$

$$C = (1+\nu)/(3+\nu)$$

(MP):

$$K_I = \lim_{r\to 0}\sqrt{2\pi}r^{1-\lambda}\sigma_{\theta\theta}(r,0) \tag{17}$$

$$K_{II} = \lim_{r\to 0}\sqrt{2\pi}r^{1-\gamma}\sigma_{r\theta}(r,0) \tag{18}$$

In eq. (16), when the V-notch singular stress field moves to the case of a crack, the stress fields should be coincident with those of the plane problem under the uniform tension, therefore the constant C must multiply for that consistency. Whereas, in eqs. (18), there are no contradiction in those treatments and no obstacls in the analytical development after that.

3.3 Singular stress fields in the vicinity of sharp V-notch

Applying the eigenfunction expansion method to the present analysis[6), 7)], we obtain

(KP):

$$\sigma_{rr} = \frac{K_{I,\lambda_1}}{\sqrt{2\pi}r^{1-\lambda_1}}\cdot\frac{1}{C_1}\left[\{-(1+3\nu)-\lambda_1(1-\nu)\}\cos(\lambda_1-1)\theta + \right.$$
$$\left. +(1-\nu)\{\kappa\cos 2\alpha(\lambda_1-1)+1+(\lambda_1-1)\cos 2\alpha\}\cos(\lambda_1+1)\theta\right] +$$
$$+\frac{K_{I,\gamma_1}}{\sqrt{2\pi}r^{1-\gamma_1}}\cdot\frac{1}{(1-\nu)C_2}\left[\{1+3\nu+\gamma_1(1-\nu)\}\sin(\gamma_1-1)\theta + \right.$$
$$\left. +(1-\nu)\{\kappa\cos 2\alpha(\gamma_1-1)-1-(\gamma_1-1)\cos 2\alpha\}\sin(\gamma_1+1)\theta\right] \tag{19}$$

$$\sigma_{\theta\theta} = \frac{K_{I,\lambda_1}}{\sqrt{2\pi}r^{1-\lambda_1}}\cdot\frac{1}{C_1}\left[\{-3-\nu+\lambda_1(1-\nu)\}\cos(\lambda_1-1)\theta - \right.$$
$$\left. -(1-\nu)\{\kappa\cos 2\alpha(\lambda_1-1)+1+(\lambda_1-1)\cos 2\alpha\}\cos(\lambda_1+1)\theta\right] +$$
$$+\frac{K_{I,\gamma_1}}{\sqrt{2\pi}r^{1-\gamma_1}}\cdot\frac{1}{(1-\nu)C_2}\left[\{1+3\nu+\gamma_1(1-\nu)\}\sin(\gamma_1-1)\theta + \right.$$
$$\left. +(1-\nu)\{\kappa\cos 2\alpha(\gamma_1-1)-1-(\gamma_1-1)\cos 2\alpha\}\sin(\gamma_1+1)\theta\right] \tag{20}$$

$$\sigma_{r\theta} = \frac{K_{I,\lambda_1}}{\sqrt{2\pi}r^{1-\lambda_1}}\cdot\frac{(1-\nu)}{C_1}\left[(\lambda_1-1)\sin(\lambda_1-1)\theta - \right.$$
$$\left. -\{\kappa\cos 2\alpha(\lambda_1-1)+1+(\lambda_1-1)\cos 2\alpha\}\sin(\lambda_1+1)\theta\right]$$
$$+\frac{K_{I,\gamma_1}}{\sqrt{2\pi}r^{1-\gamma_1}}\cdot\frac{1}{C_2}\left[(\gamma_1-1)\cos(\gamma_1-1)\theta + \right.$$
$$\left. +\{\kappa\cos 2\alpha(\gamma_1-1)-1-(\gamma_1-1)\cos 2\alpha\}\cos(\gamma_1+1)\theta\right] \tag{21}$$

where

$$C_1 = \{-3-\nu+\lambda_1(1-\nu)\}-(1-\nu)\times$$
$$\times\{\kappa\cos 2\alpha(\lambda_1-1)+1+(\lambda_1-1)\cos 2\alpha\}$$
$$C_2 = \gamma_1 - 1 +$$
$$+\{\kappa\cos 2\alpha(\gamma_1-1)-1-(\gamma_1-1)\cos 2\alpha\}$$

(22), (23)

(MP):

$$\sigma_{rr} = \frac{K_{I,\lambda_1}}{\sqrt{2\pi}r^{1-\lambda_1}}\cdot\frac{1}{C_1+1}\left[-\Lambda_1\cos(\lambda_1-1)\theta - C_1\cos(\lambda_1+1)\theta\right] +$$
$$+\frac{K_{II,\gamma_1}}{\sqrt{2\pi}r^{1-\gamma_1}}\cdot\frac{1}{-(C_2+\Gamma_2)}\left[-\Gamma_1\sin(\lambda_1-1)\theta - C_2\sin(\gamma_1+1)\theta\right] \tag{24}$$

$$\sigma_{\theta\theta} = \frac{K_{I,\lambda_1}}{\sqrt{2\pi}r^{1-\lambda_1}}\cdot\frac{1}{C_1+1}\left[\cos(\lambda_1-1)\theta + C_1\cos(\lambda_1+1)\theta\right] +$$
$$+\frac{K_{II,\gamma_1}}{\sqrt{2\pi}r^{1-\gamma_1}}\cdot\frac{1}{C_2+\Gamma_2)}\left[-\sin(\lambda_1-1)\theta - C_2\sin(\gamma_1+1)\theta\right] \tag{25}$$

$$\sigma_{r\theta} = \frac{K_{I,\lambda_1}}{\sqrt{2\pi}r^{1-\lambda_1}}\cdot\frac{2}{C_1+1}\left[\Lambda_2\sin(\lambda_1-1)\theta + C_1\sin(\lambda_1+1)\theta\right] +$$
$$+\frac{K_{II,\gamma_1}}{\sqrt{2\pi}r^{1-\gamma_1}}\cdot\frac{2}{C_2+\Gamma_2}\left[\Gamma_2\cos(\gamma_1-1)\theta + C_2\cos(\gamma_1+1)\theta\right] \tag{26}$$

where

$$C_1 = -\frac{\cos(\lambda_1-1)\alpha}{\cos(\lambda_1+1)\alpha},\quad C_2 = -\frac{\sin(\gamma_1-1)\alpha}{\sin(\gamma_1+1)\alpha}$$

$$\Lambda_1 = \frac{\lambda_1-3}{\lambda_1+1},\quad \Lambda_2 = \frac{\lambda_1-1}{\lambda_1+1}$$

$$\Gamma_1 = \frac{\gamma_1 - 3}{\gamma_1 + 1},\ \Gamma_2 = \frac{\gamma_1 - 1}{\gamma_1 + 1} \tag{27}$$

From eq. (19) ~ eq. (26), the singular stress expressions of the (KP) and the (MP) are similar in the form. For example, the case of σ_{rr} is,

$$\sigma_{rr} = \frac{K_{I,\lambda_1}}{\sqrt{2\pi} r^{1-\lambda_1}} \cdot [A\cos(\lambda_1 - 1)\theta + B\cos(\lambda_1 + 1)\theta] + \\ + \frac{K_{I,\gamma_1}}{\sqrt{2\pi} r^{1-\gamma_1}} [C\sin(\lambda_1 - 1)\theta + D\sin(\lambda_1 + 1)\theta] \tag{28}$$

where $A,\cdots,D$ are constants given by the eqs. (19) and eq. (24). However, their contents are quite different between the (KP) and the (MP), and also for the order of stress singularity λ, γ as shown in eq. (19) ~ eq. (21). Derivation of the eq. (19) ~ eq. (21) is given by the reference[6)], and that of eq. (24) ~eq. (26) is given by Apendix C.

3.4 Limiting case of a crack between the results of the two plate theory

The special case of the sharp V-notch is a crack. Therefore, as the limiting case, the singular stress fields of the sharp V-notch must be identical with those of the singular stress fields of the crack. Moreover, when a plate subjected to the uniform transverse bending, the surface of the plate becomes to the plane stress situation. Are the singular stress fields obtained here identical with the case of the plane stress fields? and the case of crack? These circumstances will be discussed in this section.

(1) Are the singular stress fields of the sharp V-notched plate coincident with those of the plane problem?

(KP), (MP):

Unfortunately, as is seen from eqs. (19) ~ (21) and eqs.(24)~(26) the both results are not identical with the plane problems[9)].

(2) The limiting case from the V-notch to a crack[6), 7)]

(KP):

$$\sigma_{rr} = -\frac{K_I}{\sqrt{2\pi r}} \cdot \frac{7+\nu}{4(3+\nu)} \left[\cos\frac{3\theta}{2} - \frac{3+5\nu}{7+\nu}\cos\frac{\theta}{2}\right] + \\ + \frac{K_{II}}{\sqrt{2\pi r}} \cdot \frac{5+3\nu}{4(1+\nu)} \left[\sin\frac{3\theta}{2} - \frac{3+5\nu}{5+3\nu}\sin\frac{\theta}{2}\right] \tag{29}$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}} \cdot \frac{7+\nu}{4(3+\nu)} \left[\cos\frac{3\theta}{2} + \frac{5+3\nu}{7+\nu}\cos\frac{\theta}{2}\right] + \\ - \frac{K_{II}}{\sqrt{2\pi r}} \cdot \frac{5+3\nu}{4(1+\nu)} \left[\sin\frac{3\theta}{2} + \sin\frac{\theta}{2}\right] \tag{30}$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \cdot \frac{7+\nu}{4(3+\nu)} \left[\sin\frac{3\theta}{2} - \frac{1-\nu}{7+\nu}\sin\frac{\theta}{2}\right] + \\ + \frac{K_{II}}{\sqrt{2\pi r}} \cdot \frac{5+3\nu}{4(1+\nu)} \left[\cos\frac{3\theta}{2} - \frac{1-\nu}{5+3\nu}\cos\frac{\theta}{2}\right] \tag{31}$$

(MP):

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}} \left[-\frac{1}{4}\cos\frac{3\theta}{2} + \frac{5}{4}\cos\frac{\theta}{2}\right] + \\ + \frac{K_{II}}{\sqrt{2\pi r}} \left[\frac{3}{4}\sin\frac{3\theta}{2} - \frac{5}{4}\sin\frac{\theta}{2}\right] \tag{32}$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}} \left[\frac{1}{4}\cos\frac{3\theta}{2} + \frac{3}{4}\cos\frac{\theta}{2}\right] + \\ + \frac{K_{II}}{\sqrt{2\pi r}} \left[-\frac{3}{4}\sin\frac{3\theta}{2} - \frac{3}{4}\sin\frac{\theta}{2}\right] \tag{33}$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \left[\frac{1}{4}\sin\frac{3\theta}{2} + \frac{1}{4}\sin\frac{\theta}{2}\right] + \\ + \frac{K_{II}}{\sqrt{2\pi r}} \left[\frac{3}{4}\cos\frac{3\theta}{2} + \frac{1}{4}\cos\frac{\theta}{2}\right] \tag{34}$$

First of all, from eqs. (29) ~ (34), the both singular stress fields are different from each other in details except the order of stress singularity. In eqs. (29) ~ (31), the generalized stress intensity factors GSIFs are defined by the eqs. (15) ~ (18). Note again the constant C in Eq. (16), and also the singular stress fields of the (KP) are influenced by the Poisson's ratio. On the contrary, eqs. (32) ~ (34) are not influenced by the Poisson's ratio. The more important situation is that the singular stress fields of eqs. (32) ~ (34) are identical with those of the plane problem[9)].

4. Conclusion

To apply the fracture mechanics to the plate bending problems, it is necessary to clarify the singular stress fields of cracks and/or sharp notches in each plate theory, and to use effectively with confirming the properties of their plate theories. In the present paper, the two singular stress fields of

Table 1 Comparison of the V-notched singular stress fields between the Kirchhoff and Mindlin plate theories

No	Category	Kirchhoff Plates	Mindlin Plates	Results
1	Characteristic eq.	Eq.(10), (11)	Eq.(12), (13)	Contradiction
2	Definition of G.S.I.Fs	Eq.(15), (16)	Eq.(17), (18)	Partly contradiction
3	Singular stress fields of Sharp V-notch	Eq.(19), (20), (21)	Eq.(24), (25) ,(26)	Contradiction
4	Comparison No.3 with plane problem	Disagreement with the plane problem	Disagreement with the plane problem	
5	Limiting value of No.3 to a crack case	Eq.(29), (30), (31)	Eq.(32), (33), (34)	Contradiction
6	Comparison No.5 with plane problem	Disagreement with the plane problem	Agreement with the plane problem[9)]	

the sharp V-notched plate based on the both Kirchhoff and the Mindlin plate theories were compered and examined. The relationships between the two plate theories, and the so-called two planes problems were compared. The results show in Table 1.

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Appendix

A

The basic equations of the (KP) and the (MP) are expressed in the polar coordinates as [10),11)]:

(KP):

$$\frac{\partial^4 w}{\partial r^4}+\frac{1}{r^3}\frac{\partial w}{\partial r}-\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}+\frac{2}{r}\frac{\partial^3 w}{\partial r^3}+\frac{4}{r^4}\frac{\partial^2 w}{\partial \theta^2}-$$
$$-\frac{2}{r^3}\frac{\partial^3 w}{\partial r\partial\theta^2}+\frac{2}{r^2}\frac{\partial^4 w}{\partial r^2\partial\theta^2}+\frac{1}{r^4}\frac{\partial^4 w}{\partial\theta^4}=q \qquad \text{(a1)}$$

(MP):

$$\frac{D}{2}\left[(1-\nu)\left\{\frac{\partial^2\psi_r}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi_r}{\partial\theta^2}-\frac{1}{r^2}\psi_r-\frac{2}{r^2}\frac{\partial\psi_\theta}{\partial\theta}\right\}+\right.$$
$$\left.+(1+\nu)\left\{\frac{\partial^2\psi_r}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_r}{\partial r}+\frac{1}{r}\frac{\partial^2\psi_\theta}{\partial r\partial\theta}-\frac{\psi_r}{r^2}-\frac{1}{r^2}\frac{\partial\psi_\theta}{\partial\theta}\right\}\right]-$$
$$-\kappa^2 Gh\left(\psi_r+\frac{\partial w}{\partial r}\right)=0 \qquad \text{(a2)}$$

$$\frac{D}{2}\left[(1-\nu)\left\{\frac{\partial^2\psi_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_\theta}{\partial r}-\frac{1}{r^2}\psi_\theta+\frac{1}{r^2}\frac{\partial^2\psi_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial\psi_r}{\partial\theta}\right\}+\right.$$
$$\left.+(1+\nu)\left\{\frac{1}{r^2}\frac{\partial\psi_\theta}{\partial\theta}+\frac{1}{r}\frac{\partial^2\psi_r}{\partial r\partial\theta}+\frac{1}{r^2}\frac{\partial\psi_r}{\partial\theta}\right\}\right]-$$
$$-\kappa^2 Gh\left(\psi_\theta+\frac{1}{r}\frac{\partial w}{\partial\theta}\right)=0 \qquad \text{(a3)}$$

$$\kappa^2 Gh\left\{\frac{\partial\psi_r}{\partial r}+\frac{1}{r}\psi_r+\frac{1}{r}\frac{\partial\psi_\theta}{\partial\theta}+\frac{\partial^2 w}{\partial r^2}+\right.$$
$$\left.+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right\}+q=0 \qquad \text{(a4)}$$

B

We assume the generalized displacements of (MP) $\psi_r(r,\theta)$, $\psi_\theta(r,\theta)$, $w(r,\theta)$ in the eqs. (a2), (a3) and (a4) as:

$$\psi_r(r,\theta)=r^{\lambda}\Psi_r(\theta) \qquad \text{(a5)}$$
$$\psi_\theta(r,\theta)=r^{\lambda}\Psi_\theta(\theta) \qquad \text{(a6)}$$
$$w(r,\theta)=r^{\lambda'+1}W(\theta) \qquad \text{(a7)}$$

where λ, λ' are the eigenvalues and they are complex in general. Substituting eqs. (a5), (a6), (a7) into eqs. (a2), (a3), (a4), we obtain

$$D\left[(1-\nu)\left\{(\lambda^2-1)\Psi_r(\theta)-2\Psi_\theta'(\theta)+\Psi_r''(\theta)\right\}+(1+\nu)\times\right.$$
$$\left.\times\left\{(\lambda^2-1)\Psi_r(\theta)+(\lambda-1)\Psi_\theta'(\theta)\right\}\right]-$$
$$-2\kappa^2 Gh\left[r^2\Psi_r(\theta)+(\lambda'+1)r^{\bar{\lambda}-\lambda+2}W(\theta)\right]=0 \qquad \text{(a8)}$$

$$D\left[(1-\nu)\left\{(\lambda^2-1)\Psi_\theta(\theta)+2\Psi_r'(\theta)+\Psi_\theta''(\theta)\right\}+(1+\nu)\times\right.$$
$$\left.\times\left\{(\lambda+1)\Psi_r'(\theta)+\Psi_\theta''(\theta)\right\}\right]-$$
$$-2\kappa^2 Gh\left[r^2\Psi_\theta(\theta)+r^{\bar{\lambda}-\lambda+2}W'(\theta)\right]=0 \qquad \text{(a9)}$$

$$W''(\theta)+(\lambda'+1)^2 W(\theta)+r^{\lambda-\lambda'}\{(\lambda+1)\Psi_r(\theta)+\Psi_\theta'(\theta)\}=0 \quad \text{(a10)}$$

The displacements at the vertex must be finite when $r \to 0$, and from eqs. (a8), (a9) and (a10)

$$\begin{aligned}&[(1-\nu)\{(\lambda^2-1)\Psi_r(\theta)-2\Psi_\theta'(\theta)+\Psi_r''(\theta)\}+\\&+(1+\nu)\{(\lambda^2-1)\Psi_r(\theta)+(\lambda-1)\Psi_\theta'(\theta)\}]=0\end{aligned} \quad \text{(a11)}$$

$$\begin{aligned}&[(1-\nu)\{(\lambda^2-1)\Psi_\theta(\theta)+2\Psi_r'(\theta)+\Psi_\theta''(\theta)\}+\\&+(1+\nu)\{(\lambda+1)\Psi_r'(\theta)+\Psi_\theta''(\theta)\}]=0\end{aligned} \quad \text{(a12)}$$

$$W''(\theta)+(\lambda'+1)^2 W(\theta)=0 \quad \text{(a13)}$$

From eq.(a12), we obtain

$$\begin{aligned}&\{\lambda(1+\nu)+3-\nu\}\Psi_r'(\theta)=\\&=-2\Psi_\theta''(\theta)-(1-\nu)(\lambda^2-1)\Psi_\theta(\theta)\end{aligned} \quad \text{(a14)}$$

Eliminating $\Psi_r(\theta)$ from the simultaneous partial differential equations of (a11) and (a12) give

$$\Psi_\theta^{IV}(\theta)+2(\lambda^2+1)\Psi_\theta''(\theta)+(\lambda^2-1)^2\Psi_\theta(\theta)=0 \quad \text{(a15)}$$

The general solution of eq.(a15) is

$$\begin{aligned}\Psi_\theta(\theta)&=A\cos(\lambda+1)\theta+B\sin(\lambda+1)\theta+\\&+C\cos(\lambda-1)\theta+D\sin(\lambda-1)\theta\end{aligned} \quad \text{(a16)}$$

where A, $\cdots$, D are arbitrary constants. Substituting eq. (16) into eq. (a14), we obtain the general solution of $\Psi_r(\theta)$ as

$$\begin{aligned}\Psi_r(\theta)&=A\sin(\lambda+1)\theta-B\cos(\lambda+1)\theta+\\&+\delta C\sin(\lambda-1)\theta-\delta D\cos(\lambda-1)\theta\end{aligned} \quad \text{(a17)}$$

where

$$\delta=\{\lambda(1+\nu)-3+\nu\}/\{\lambda(1+\nu)+3-\nu\} \quad \text{(a18)}$$

and from eq.(a23)

$$W(\theta)=F\sin(\lambda'+1)\theta+G\cos(\lambda'+1)\theta \quad \text{(a19)}$$

where F and G are arbitrary constants.
Thus, finally from eqs. (a5), (a6) and (a7), we obtain

$$\begin{aligned}\psi_r(r,\theta)&=r^\lambda[A\sin(\lambda+1)\theta-B\cos(\lambda+1)\theta+\\&+\delta C\sin(\lambda-1)\theta-\delta D\cos(\lambda-1)\theta]\end{aligned} \quad \text{(a20)}$$

$$\begin{aligned}\psi_\theta(r,\theta)&=r^\lambda[A\cos(\lambda+1)\theta+B\sin(\lambda+1)\theta+\\&+C\sin(\lambda-1)\theta+D\cos(\lambda-1)\theta]\end{aligned} \quad \text{(a21)}$$

$$w(\theta)=r^{\lambda'+1}[F\sin(\lambda'+1)\theta+G\cos(\lambda'+1)\theta] \quad \text{(a22)}$$

Relationships between the moments ($M_{\theta\theta}$, $M_{r\theta}$) the shearing force Q_θ and the generalized displacements of eqs. (a20), (a21) and (a22) are [12)]

$$M_{rr}=D\left[\frac{\partial\psi_r}{\partial r}+\frac{\nu}{r}\left(\psi_r+\frac{\partial\psi_\theta}{\partial\theta}\right)\right] \quad \text{(a23)}$$

$$M_{\theta\theta}=D\left[\frac{1}{r}\left(\psi_r+\frac{\partial\psi_\theta}{\partial\theta}\right)+\nu\frac{\partial\psi_r}{\partial r}\right] \quad \text{(a24)}$$

$$M_{r\theta}=\frac{D(1-\nu)}{2}\left[\frac{\partial\psi_\theta}{\partial r}+\frac{1}{r}\frac{\partial\psi_r}{\partial\theta}-\frac{1}{r}\psi_\theta\right] \quad \text{(a25)}$$

$$Q_r=\kappa^2 Gh\left(\psi_r+\frac{\partial w}{\partial r}\right) \quad \text{(a26)}$$

$$Q_\theta=\kappa^2 Gh\left(\psi_\theta+\frac{1}{r}\frac{\partial w}{\partial\theta}\right) \quad \text{(a27)}$$

Substituting eqs. (a20) ~ (a22) into eqs. (a24), (a25) and (a27), and taking into consideration of boundary conditions eq. (7) ~ eq. (9), we obtain

$$\begin{aligned}&B\lambda(1-\nu)\cos(\lambda+1)\alpha+\\&+D(-\delta-\nu\lambda\delta+\lambda-1)\cos(\lambda-1)\alpha=0\end{aligned} \quad \text{(a28)}$$

$$\begin{aligned}&A\lambda(1-\nu)\sin(\lambda+1)\alpha-\\&-C(\delta+\nu\lambda\delta-\lambda+1)\sin(\lambda-1)\alpha=0\end{aligned} \quad \text{(a29)}$$

$$\begin{aligned}&2A\lambda\cos(\lambda+1)\alpha+\\&+C\{\lambda+\delta(\lambda-1)-1\}\cos(\lambda-1)\alpha=0\end{aligned} \quad \text{(a30)}$$

$$\begin{aligned}&2B\lambda\sin(\lambda+1)\alpha+\\&+D\{\lambda+\delta(\lambda-1)-1\}\sin(\lambda-1)\alpha=0\end{aligned} \quad \text{(a31)}$$

$$F\cos(\lambda'+1)\alpha-G\sin(\lambda'+1)\alpha=0 \quad \text{(a32)}$$

$$F\cos(\lambda'+1)\alpha+G\sin(\lambda'+1)\alpha=0 \quad \text{(a33)}$$

For the non-trivial solution of the set of eqs. (a28) ~ (31), and eqs. (32) and (33), the determinant of the coefficients of the unknowns must be equal to zero. Thus, the characteristic equations to be determined the eigenvalues of λ, and λ' are

$$(\lambda\sin 2\alpha+\sin 2\lambda\alpha)(\lambda\sin 2\alpha-\sin 2\lambda\alpha)=0 \quad \text{(a34)}$$

or

$$\lambda\sin 2\alpha+\sin 2\lambda\alpha=0 \quad \text{(a35)}$$

$$\lambda\sin 2\alpha-\sin 2\lambda\alpha=0 \quad \text{(a36)}$$

and similarly from eqs. (a32) and (a33),

$$\sin 2(\lambda' + 1)\alpha = 0 \qquad \text{(a37)}$$

Thus, the characteristic eqs. (12) ~ (14) were derived.

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Eqs. (a35) and (a36) correspond to the mode I and mode II, respectively. Eq.(a37) corresponds to the out of plane shear.

Using eqs.(a28)~ (a31), eqs. (a20) ~ (a22) can be written in the eigenfunction expansions as follows:

$$\psi_r(r,\theta) = \sum_{k=1}^{\infty}\left[D_k r^{\lambda_k}\{\eta_k \cos(\lambda_k + 1)\theta - -\delta_k \cos(\lambda_k - 1)\theta\} + + C_k r^{\gamma_k}\{-\mu_k \sin(\gamma_k + 1)\theta + \delta'_k \sin(\gamma_k - 1)\}\right] \qquad \text{(a38)}$$

$$\psi_\theta(r,\theta) = \sum_{k=1}^{\infty}\left[D_k r^{\lambda_k}\{-\eta_k \sin(\lambda_k + 1)\theta + -\delta_k \sin(\lambda_k - 1)\theta\} + + C_k r^{\gamma_k}\{-\mu_k \cos(\gamma_k + 1)\theta + \cos(\gamma_k - 1)\theta\}\right] \qquad \text{(a39)}$$

$$w(r,\theta) = \sum_{k=1}^{\infty} r^{\lambda_k + 1}\{G_k \cos(\lambda'_k + 1)\theta + + F_k \sin(\lambda'_k + 1)\} \qquad \text{(a40)}$$

where

$$\left.\begin{aligned} \eta_k &= \frac{(\lambda_k + 1)(1+\nu)}{\lambda_k(1+\nu) + 3 - \nu}\cdot\frac{\cos(\lambda_k - 1)\alpha}{\cos(\lambda_k + 1)\alpha} \\ \mu_k &= \frac{(\gamma_k + 1)(1+\nu)}{\gamma_k(1+\nu) + 3 - \nu}\cdot\frac{\cos(\gamma_k - 1)\alpha}{\cos(\gamma_k + 1)\alpha} \\ \delta_k &= \frac{\lambda_k(1+\nu) - 3 + \nu}{\lambda_k(1+\nu) + 3 - \nu} \\ \delta'_k &= \frac{\gamma_k(1+\nu) - 3 + \nu}{\gamma_k(1+\nu) + 3 - \nu} \end{aligned}\right\} \qquad \text{(a41)}$$

and, λ_k and γ_k are characteristic values of mode I and mode II, respectively.

The component of the moments are derived from eqs. (a23)~(a25) and eqs. (a38) and (a39). Using stress- displacement equations of

$$\varepsilon_{rr} = z\frac{\partial \psi_r}{\partial r} \qquad \text{(a42)}$$

$$\varepsilon_{\theta\theta} = \frac{z}{r}\left(\psi_r + \frac{\partial \psi_\theta}{\partial \theta}\right) \qquad \text{(a43)}$$

$$\varepsilon_{r\theta} = \frac{z}{2}\left(\frac{1}{r}\frac{\partial \psi_r}{\partial \theta} + \frac{\partial \psi_\theta}{\partial r} - \frac{\psi_\theta}{r}\right) \qquad \text{(a44)}$$

under the plane stress condition, thus, the stress components are given by

$$\frac{(1-\nu^2)}{E}\sigma_{rr} = z\left\{\frac{\partial \psi_r}{\partial r} + \frac{\nu}{r}\left(\psi_r + \frac{\partial \psi_\theta}{\partial \theta}\right)\right\} \qquad \text{(a45)}$$

$$\frac{(1-\nu^2)}{E}\sigma_{\theta\theta} = z\left\{\frac{1}{r}\left(\psi_r + \frac{\partial \psi_\theta}{\partial \theta}\right) + \nu\frac{\partial \psi_r}{\partial r}\right\} \qquad \text{(a46)}$$

$$\frac{(1-\nu)}{E}\sigma_{r\theta} = \frac{z}{2}\left(\frac{1}{r}\frac{\partial \psi_r}{\partial \theta} + \frac{\partial \psi_\theta}{\partial r} - \frac{\psi_\theta}{r}\right) \qquad \text{(a47)}$$

where z is the distance from the midplane in the direction of the z axis, and the other stress components are omitted here. Substituting eqs. (a38) and (a39) into eqs. (a45) ~ (a47), and putting k=1, we obtain eqs. (24) ~ (26) in the section (3.3).

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