Paper

Alternative Complex Representations in Anisotropic Thermoelastic Media and Related Properties

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In this paper, Green and Zerna's complex potential approach[1)], which is the theory of isothermal anisotropic elasticity, is extended to the anisotropic thermoelasticity. First, based on the approach, the stress and the displacement components are expressed by the complex potential functions and the thermal potential function. Special attention is paid to the transformation formulas of the complex potentials and the physical constants attendant on the translation and rotation of the rectangular coordinate systems. These formulas are correspond to the case of Muskhelishvili's transformation formulas[2)] for the isotropic, isothermal body. These will be very convenient to construct the three complex potentials in the case of using multiple rectangular coordinates for the various boundary value problems of anisotropic thermoelasticity. Finally, a few fundamental solutions are given by the complex potential representation obtained.

Key Words: *anisotropic thermoelasticity, complex potential functions, translation and rotation of coordinates, fundamental complex potential functions*

1. Introduction

In the mid-nineteenth century, Muskhelishvili[2)] considerably extended the complex variable method of elasticity by adding the idea of Cauchy integrals and conformal mapping, and solved a large number of specific problems summarized in his very famous book[2)]. It can be said that he lastly systematized the two dimensional, isotropic complex variable method of elasticity. One of the advantages of the method is that it is of very convenience in treatment of stress singularities in elastic solids, and was applied to the various fracture mechanics analyses for isotropic bodies. Subsequently, the method was extended by Lekhnitukii[3)], Eshelby et. al[4)], and Stroh[5)] to the anisotropic elastic body, independently. They are well known as Lekhnitskii formalism and Eshelby-Stroh formalism, respectively. Moreover, these formalisms were extended to the anisotropic thermoelasticity[6)-9)], the piezoelectric elastic solids[10),11)] and others[12)].

On the other hand, there is the Green and Zerna formalism[1)] as an alternative formalism, which was independently developed by connecting tensor analyses and complex variables. Based on this formalism, the author has studied on the fracture analyses associated with singular stress fields such as crack problems[13)], flat inclusion problems[14)] and stiffener problems[15)]. This formalism is useful for construction of complex potential functions in the case of employing the multiple coordinate systems, similar to the transformation formulas[2)] of Muskhelishvili's complex potentials.

In the present paper, the basic analyses are made for extending the previous fracture analyses[13)-15)] of anisotropic isothermal body to the anisotropic thermoelastic body. In order to apply the method directly to the anisotropic thermoelastic body, first of all we derived the basic equations of the body in terms of complex variables, and found the transformation formulas of the complex potential functions, and the physical constants and the induced constants. Finally, although well known simple solutions, we derived the solutions of the temperature dislocation and the concentrated heat source in an infinite thermoelastic body with our complex potentials.

2. Basic Equations

First, complex potential representations of basic equations of two dimensional, anisotropic thermo- elasticity are derived.

In the absence of body forces, stress components $\tau^{\alpha\beta}$ can be expressed in terms of Airy's stress function U in general coordinates by the formula

$$\tau^{\alpha\beta} = \varepsilon^{\gamma\alpha}\varepsilon^{\rho\beta}U\big|_{\rho\gamma} \tag{1}$$

where $\varepsilon^{\alpha\beta}$ is ε-system of order two and $U\big|_{\alpha\beta}$ denotes covariant differentiation of the stress function U. From now on, Greek indices denote 1 or 2. The strain tensor $\gamma^{\alpha\beta}$ is related to the displacement tensor $v^{\alpha\beta}$ as follows:

$$\gamma^{\alpha\beta} = \frac{1}{2}\left(a^{\alpha\lambda}v^{\beta}\big|_{\lambda} + a^{\beta\lambda}v^{\alpha}\big|_{\lambda}\right) \tag{2}$$

where $a^{\alpha\beta}$ is a metric tensor. The generalized Hooke's law by referring the stress and strain to the two-dimensional general coordinates is expressed by the general form

$$\gamma^{\alpha\beta} = F^{\alpha\beta}_{\lambda\mu}\tau^{\lambda\mu} + G^{\alpha\beta}T \tag{3}$$

where $F^{\alpha\beta}_{\lambda\mu}$ denotes the elastic constant and $G^{\alpha\beta}$ corresponds to the thermal expansion coefficient at temperature T. The equation of conduction of heat in the steady state is given by

$$M^{\alpha\beta}T\big|_{\alpha\beta} = 0 \tag{4}$$

where $M^{\alpha\beta}$ is the conductivity coefficient of the medium. Also, the heat flux f^{α} is given by

$$f^{\alpha} = -\kappa^{\alpha\beta}T\big|_{\beta} \tag{5}$$

where $\kappa^{\alpha\beta}$ is the thermal conductivity constant. Equations (1)-(5) above are the basic equations of homogeneous anisotropic, thermoelastic bodies.

3. Representation of the governing equations in terms of complex coordinates

If we denote covariant and contravariant base vectors in the complex coordinate systems by $\mathbf{a}_{\alpha}$ and $\mathbf{a}^{\alpha}$ respectively, the position vector $\mathbf{r}$ may be written as

$$\mathbf{r} = \mathbf{z}^{\alpha}\mathbf{a}_{\alpha} = \mathbf{z}_{\alpha}\mathbf{a}^{\alpha} \tag{6}$$

where the complex coordinates $(z, \bar{z})$ are introduced by the formulas

$$z = x + iy,\ \bar{z} = x - iy \tag{7}$$

and noting that $x = x_1$, $y = x_2$, $z^1 = z$, $z^2 = \bar{z}$ and $z^3 = x_3$ by tensor transformations. We now restrict our attention to bodies which are elastically symmetrical with respect to the plane (x_1, x_2) and assume the state of plane strain (Appendix A).

The tensors $D^{\alpha\beta}$, $T^{\alpha\beta}$ and F^{α} which can be expressed by the displacement u^{α}, the stress $t^{\alpha\beta}$, and the heat flux f^{α} in the rectangular coordinates x_{α}, are introduced as follows:

$$D^{\alpha}=\frac{\partial z^{\alpha}}{\partial x^{\beta}}u^{\beta} \quad (8)$$

$$T^{\alpha\beta}=\frac{\partial z^{\alpha}}{\partial x^{\gamma}}\frac{\partial z^{\beta}}{\partial x^{\delta}}t^{\gamma\delta} \quad (9)$$

$$F^{\alpha}=\frac{\partial z^{\alpha}}{\partial x^{\beta}}f^{\beta} \quad (10)$$

From Eqs.(8),(9) and (10), we have the following expressions, which need the later analyses:

$$T^{12}=t^{11}+t^{22}=t_{11}+t_{22}\equiv\Theta=\sigma_{xx}+\sigma_{yy} \quad (11)$$

$$T^{11}=t_{11}-t_{22}+2it_{12}\equiv\Phi=\sigma_{xx}-\sigma_{yy}+2i\sigma_{xy} \quad (12)$$

$$D^{1}=u_{1}+iu_{2}\equiv D=u_{x}+iu_{y} \quad (13)$$

$$F^{1}=f_{1}+if_{2}\equiv F=f_{x}+if_{y} \quad (14)$$

If we remember that $u^{\alpha}=u_{\alpha}$, $t^{\alpha\beta}=t_{\alpha\beta}$ in the rectangular coordinates and hereafter these properties may be used without notice. In Eqs.(11)-(14), we denoted $t_{\alpha\beta}, u_{\alpha}$ and f_{α}, by $\sigma_{xx},\cdots,u_{x},\cdots,f_{y}$, since there is no risk of confusion in two dimensional elasticity. To develop the theory with complex variable coordinates, it is more convenient to introduce the tensors $F^{\alpha\beta}_{\lambda\mu}, G^{\alpha\beta}$ and $M^{\alpha\beta}$ referred to complex coordinates, and they are introduced, respectively, as follows:

$$S^{\alpha\beta}_{\lambda\mu}=s^{\rho\gamma}_{\xi\eta}\frac{\partial z^{\alpha}}{\partial x^{\rho}}\frac{\partial z^{\beta}}{\partial x^{\gamma}}\frac{\partial z^{\xi}}{\partial x^{\lambda}}\frac{\partial z^{\eta}}{\partial x^{\mu}} \quad (15)$$

$$H^{\alpha\beta}=\alpha^{\gamma\delta}\frac{\partial z^{\alpha}}{\partial x^{\gamma}}\frac{\partial z^{\beta}}{\partial x^{\delta}} \quad (16)$$

$$K^{\alpha\beta}=\kappa^{\gamma\delta}\frac{\partial z^{\alpha}}{\partial x^{\gamma}}\frac{\partial z^{\beta}}{\partial x^{\delta}} \quad (17)$$

where $s^{\rho\gamma}_{\xi\eta}$, $\alpha^{\gamma\delta}$, $\kappa^{\gamma\delta}$ are the elastic constant, the thermal expansion coefficient and the heat conductivity in the rectangular coordinates, respectively. From Eq.(15), and taking account of a elastically symmetrical plane stated above, we see the elastic coefficient $s^{\rho\gamma}_{\xi\eta}$ reduced from 21 to the 13 independent coefficients:

$$\left.\begin{aligned}
&S^{11}_{22}=\overline{S}^{22}_{11}=(s^{11}_{11}+s^{22}_{22}-4s^{12}_{12}-2s^{11}_{22}+4is^{11}_{12}-4is^{12}_{22})/4=\\
&=(s_{11}+s_{22}-s_{66}-2s_{12}+2is_{16}-2is_{26})/4=S_{1}\\
&S^{12}_{22}=S^{11}_{12}=S_{2}=(s_{11}-s_{22}+is_{16}+is_{26})/4\\
&S^{11}_{11}=S^{22}_{22}=S_{3}=(s_{11}+s_{22}+s_{66}-2s_{12})/4\\
&S^{12}_{12}=S_{4}=(s_{11}+s_{22}+2s_{12})/4
\end{aligned}\right\} \quad (18)$$

and

$$S^{12}_{22}=S^{11}_{12}=\overline{S}^{12}_{11}=\overline{S}^{22}_{11}=S_{2} \quad (19)$$

where the generalized Hooke's law in engineering representation are given as follows:

$$\left.\begin{aligned}
\varepsilon_{11}&=\varepsilon_{xx}=s_{11}\sigma_{xx}+s_{12}\sigma_{xy}+s_{16}\sigma_{xy}+\alpha_{11}T\\
\varepsilon_{22}&=\varepsilon_{yy}=s_{11}\sigma_{xx}+s_{12}\sigma_{xy}+s_{26}\sigma_{xy}+\alpha_{22}T\\
\varepsilon_{12}&=\varepsilon_{xy}=s_{11}\sigma_{xx}+s_{12}\sigma_{xy}+s_{66}\sigma_{xy}+\alpha_{12}T
\end{aligned}\right\} \quad (20)$$

From Eq.(16) and (17), after similar manipulations, we obtain

$$H_{11}=\alpha_{11}-\alpha_{22}+2i\alpha_{12}=\overline{H}_{22},\ H_{11}=\alpha_{11}+\alpha_{22}=\overline{H}_{12} \quad (21)$$

$$K_{11}=\kappa_{11}-\kappa_{22}+2i\kappa_{12}=\overline{K}_{22},\ K_{12}=\kappa_{11}+\kappa_{22}=\overline{K}_{12} \quad (22)$$

where we also used the fact that there are no difference between the covariant and contravariant tensor in rectangular coordinates. From Eqs.(18) to (21), it should be noted that the very simple formulas for rotation of the coordinates hold, which will be shown later.

The metric tensor which corresponds to the complex coordinates Eq.(6) has constant components so that the covariant differentiation in this coordinate system reduce to partial differentiation. Thus from Eq.(1) we obtain

$$\tau_{11} = \Phi = -4\frac{\partial^2 U}{\partial \bar{z}\partial \bar{z}},\ \tau_{12} = \Theta = -4\frac{\partial^2 U}{\partial z\partial \bar{z}} \tag{23}$$

With Eqs.(2) and (3), these give (Appendix B)

$$S_3\Phi + S_1\bar{\Phi} + 2S_2\Theta = 2\frac{\partial D}{\partial \bar{z}} + H_{11}T \tag{24}$$

$$\bar{S}_1\Phi + S_3\bar{\Phi} + 2\bar{S}_2\Theta = 2\frac{\partial \bar{D}}{\partial z} + H_{22}T \tag{25}$$

$$\bar{S}_2\Phi + S_2\bar{\Phi} + 2S_4\Theta = \frac{\partial D}{\partial z} + \frac{\partial \bar{D}}{\partial \bar{z}} + H_{12}T \tag{26}$$

Eliminating D from Eq.(24)~(26) gives

$$S_1\frac{\partial^4 U}{\partial z^4} - 4S_2\frac{\partial^4 U}{\partial z^3\partial \bar{z}} + 2(S_3+S_4)\frac{\partial^4 U}{\partial z^2\partial \bar{z}^2} - 4\bar{S}_2\frac{\partial^4 U}{\partial z\partial \bar{z}^3} + $$
$$+ S_1\frac{\partial^4 U}{\partial \bar{z}^4} = -\frac{1}{4}(H_{11}\frac{\partial^2 T}{\partial z^2} - 2H_{12}\frac{\partial^2 T}{\partial z\partial \bar{z}} + H_{22}\frac{\partial^2 T}{\partial \bar{z}^2}) \tag{27}$$

Moreover the heat conduction equation of two dimensional anisotropic medium given by Eqs.(4) and (17) are obtained by using the Green and Zerna's formulation as follows:

$$K_{11}\frac{\partial^2 T}{\partial z^2} + 2K_{12}\frac{\partial^2 T}{\partial z\partial \bar{z}} + K_{22}\frac{\partial^2 T}{\partial \bar{z}^2} = 0 \tag{28}$$

4. General solution

The solution of Eq.(28) is of the form:

$$T(x,y) = \theta(z_3) + \overline{\theta(z_3)} \tag{29}$$

where $\theta(z_3)$ is an analytic function and z_3 is given by

$$z_3 = z + \gamma_3\bar{z} \tag{30}$$

The characteristic value γ_3 with $|\gamma_3| \le 1$ in Eq.(30) can be obtained from

$$K_{22}(\gamma_3)^2 + 2K_{12}\gamma_3 + K_{11} = 0 \tag{31}$$

The general solution of Eq.(27) can be taken in the form:

$$U = U_h + \phi_p \tag{32}$$

where U_h is a general solution of the homogeneous equation of Eq.(27) and is given by

$$U_h = \Omega(z_1) + \overline{\Omega(z_1)} + \omega(z_2) + \overline{\omega(z_2)} \tag{33}$$

with

$$z_1 = z + \gamma_1\bar{z},\quad z_2 = z + \gamma_2\bar{z} \tag{34}$$

and $\gamma_j (j=1,2)$ is a root of the equation:

$$\bar{S}_1\gamma^4 - 4\bar{S}_2\gamma^3 + 2(S_3+S_4)\gamma^2 - S4\gamma + S_1 = 0 \tag{35}$$

and the roots can be selected to be these roots with modulus less than unity so that

$$|\gamma_1|,\ |\gamma_2| < 1$$

On the other hand, a particular solution ϕ_p can be put as

$$\phi_p = A\iint\theta_3(z_3)dz_3dz_3 + \bar{A}\iint\overline{\theta_3(z_3)}d\bar{z}_3d\bar{z}_3 \tag{36}$$

in which A is a complex constant. By substituting Eq.(33) and (36) into Eq.(27) and comparing the coefficient of both sides of the equation, we get

$$A = -\frac{H(\gamma_3)}{4S(\gamma_3)} \tag{37}$$

where

$$\left.\begin{aligned} H(\gamma_3) &= H_{11} - 2H_{12}\gamma_3 + H_{22}\gamma_3{}^2 \\ S(\gamma_3) &= \bar{S}_1\gamma_3^4 - 4\bar{S}_2\gamma_3^3 + 2(S_3+S_4)\gamma_3^2 - S_2\gamma_3 + S_1 \end{aligned}\right\} \tag{38}$$

Thus the general solution of Eq.(27) has in the form:

$$2U = \mathrm{Re}[\Omega(z_1) + \omega(z_2) + \psi(z_3)] \tag{39}$$

where an analytic function $\psi(z_3)$ is introduced as follows:

$$\psi(z_3) = -\frac{H(\gamma_3)}{4S(\gamma_3)}\iint\theta(z_3)dz_3dz_3 \tag{40}$$

Therefore, by substituting Eq.(39) into Eq.(23), and by exchanging τ^{11}, τ^{22} and τ^{12} with σ_{xx}, σ_{yy} and σ_{xy}, we obtain

$$\begin{aligned}\sigma_{xx}+\sigma_{yy}=&\frac{4\gamma_1}{(1+\gamma_1)^2}\Omega''(\zeta_1)+\frac{4\bar{\gamma}_1}{(1+\bar{\gamma}_1)^2}\overline{\Omega''(\zeta_1)}+\\&+\frac{4\gamma_2}{(1+\gamma_2)^2}\omega(\zeta_2)+\frac{4\bar{\gamma}_2}{(1+\bar{\gamma}_2)^2}\overline{\omega(\zeta_2)}+\\&+\frac{4\gamma_3}{(1+\gamma_3)^2}\psi(\zeta_3)+\frac{4\bar{\gamma}_3}{(1+\bar{\gamma}_3)^2}\overline{\psi(\zeta_3)}\end{aligned}\qquad(41)$$

$$\begin{aligned}\sigma_{xx}-\sigma_{yy}+2i\sigma_{xy}=&-\frac{4\gamma_1^2}{(1+\gamma_1)^2}\Omega''(\zeta_1)-\frac{4}{(1+\bar{\gamma}_1)^2}\overline{\Omega''(\zeta_1)}-\\&-\frac{4\gamma_2}{(1+\gamma_2)^2}\omega''(\zeta_2)-\frac{4}{(1+\bar{\gamma}_2)^2}\overline{\omega''(\zeta_2)}-\\&-\frac{4\gamma_3^2}{(1+\gamma_3)^2}\psi''(\zeta_3)-\frac{4}{(1+\bar{\gamma}_3)^2}\overline{\psi''(\zeta_3)}\end{aligned}\qquad(42)$$

and for the temperature distribution, from Eq.(29)

$$T(x,y)=\theta(\zeta_3)+\overline{\theta(\zeta_3)}\qquad(43)$$

where the three complex variables $\zeta_i\,(i=1,2,3)$ were introduced as the affine transformation as follows:

$$\zeta_j=\frac{z+\gamma_j\bar{z}}{1+\gamma_j},\ (j=1,2,3)\qquad(44)$$

Similarly the displacement components u_x and u_y are obtained by substituting Eq.(39) into Eq.(24) and integrating as follows:

$$\begin{aligned}u_x+iu_y=&\frac{\delta_1}{1+\gamma_1}\Omega'(\zeta_1)+\frac{\rho_1}{1+\bar{\gamma}_1}\overline{\Omega'(\zeta_1)}+\\&+\frac{\delta_2}{1+\gamma_2}\omega'(\zeta_2)+\frac{\delta_2}{1+\bar{\gamma}_2}\overline{\omega'(\zeta_2)}+\\&+\frac{\delta_3}{1+\gamma_3}\psi(\zeta_3)+\frac{\delta_3}{1+\bar{\gamma}_3}\overline{\psi(\zeta_3)}\end{aligned}\qquad(45)$$

where

$$\delta_j=-2(S_3\gamma_j^2-2S_2\gamma_j+S_1)/\gamma_j-\frac{4H_{11}S(\gamma_j)}{\gamma_j H(\gamma_j)},\ (j=1,2,3)\qquad(46)$$

$$\rho_j=-2(S_3-2S_2\bar{\gamma}_j+S_1\bar{\gamma}_j^2)-\frac{4H_{11}\overline{S(\gamma_j)}}{\overline{H(\gamma_j)}},\ (j=1,2,3)\qquad(47)$$

and where we must exclude the second term in the right hand sides of Eqs.(46) and (47) when j=1 and j=2. Moreover the resultant force P exerted across a part AB on the curve of the body, and the moment M about x_3 axis which perpendicular to the (x,y) plane, are found according to Green and Zerna[1] in the forms :

$$\begin{aligned}P=X+iY=2i\frac{\partial U}{\partial\bar{z}}=2i\Big[&\frac{\gamma_1}{1+\gamma_1}\Omega'(\zeta_1)+\frac{1}{1+\bar{\gamma}_1}\overline{\Omega'(\zeta_1)}+\\&+\frac{\gamma_2}{1+\gamma_2}\omega'(\zeta_2)+\frac{1}{1+\bar{\gamma}_2}\overline{\omega'(\zeta_2)}+\frac{\gamma_3}{1+\gamma_3}\psi(\zeta_3)+\frac{1}{1+\bar{\gamma}_3}\overline{\psi(\zeta_3)}\Big]_A^B\end{aligned}\qquad(48)$$

$$\begin{aligned}M=&z\frac{\partial U}{\partial z}+\bar{z}\frac{\partial U}{\partial\bar{z}}-U=\\=&\big[\zeta_1\Omega'(\zeta_1)+\bar{\zeta}_1\overline{\Omega'(\zeta_1)}-\Omega'(\zeta_1)-\overline{\Omega'(\zeta_1)}+\\&+\zeta_2\omega'(\zeta_2)+\bar{\zeta}_2\overline{\omega'(\zeta_2)}-\omega'(\zeta_2)-\overline{\omega'(\zeta_2)}+\\&+\zeta_3\psi(\zeta_3)+\zeta_3\overline{\psi(\zeta_3)}-\psi(\zeta_3)-\overline{\psi(\zeta_3)}\big]_A^B\end{aligned}\qquad(49)$$

where X and Y are the components of resultant forces in the direction of x and y axes, respectively. Thus, the basic equations of complex potential representation for two dimensional thermoelasticity of anisotropic media are given by Eqs.(41),(42),(43) and (45). It should be noted that the rigid body displacements and rotation are neglected in those equations.

5. Rectangular coordinate transformations of the complex potential functions and physical constants

5.1 Translation

Now, let us consider how the complex functions corresponding to a given stress state of a body change under the translation from one system of rectangular coordinates to another, as shown in **Fig.1**. When the new system $t(=x+iy)$ is translated from the old one $z_1(=x_1+iy_1)$ by a distance $z_{10}(=x_{10}+iy_{10})$, by the condition of stress components are not altered by a translation, the transformation relation between the complex potentials $\Omega_1(\zeta_1)$, $\omega_1(\zeta_2)$ and $\psi_1(\zeta_3)$ related to old coordinates (x_1,y_1) and the new complex potentials $\Omega_1(\zeta_1)$, $\omega_1(\zeta_2)$ and $\psi_1(\zeta_3)$ are obtained as follows:

$$\left.\begin{aligned}\Omega_1'(\zeta_1)&=\Omega'(\zeta_1-\zeta_{10})\\ \omega_1'(\zeta_2)&=\omega'(\zeta_1-\zeta_{20})\\ \psi_1'(\zeta_3)&=\psi'(\zeta_1-\zeta_{30})\end{aligned}\right\} \quad (50)$$

where

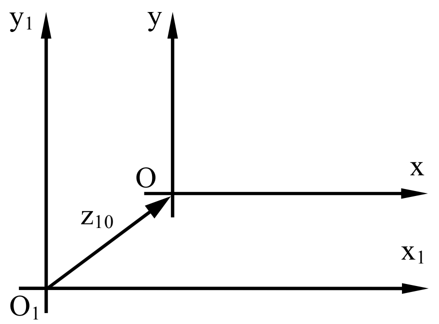


Fig.1 Translation of rectangular coordinates

$$\zeta_j=\frac{z_j}{1+\gamma_j},\quad z_j=z+\gamma_j\bar{z},\ (j=1,2,3) \quad (51)$$

$$\zeta_{j0}=\frac{z_{10}+\gamma_j\bar{z}_{10}}{1+\gamma_j},\ z_{10}=x_{10}+iy_{10},\ (j=1,2,3) \quad (52)$$

5.2 Rotation

Next consider the effect of rotating the axes θ leaving the origin fixed as shown in **Fig.2**. We denote the old coordinate systems as $(x_1,\ y_1)$ and the new systems $(x,\ y)$ whose the direction cosines relative to the old system is $c_{ij},\ (i,j=1,2)$, then the relation between the new conductivity coefficients $\kappa'_{\alpha\beta}$ of the new system $(x,\ y)$ and $\kappa_{\alpha\beta}$ of the old system $(x_1,\ y_1)$ is

$$\kappa'_{\alpha\beta}=c_{\alpha r}c_{\beta s}\kappa_{rs},\ (\alpha,\beta,r,s=1,2) \quad (53)$$

where κ_{rs} is the conductivity coefficient in

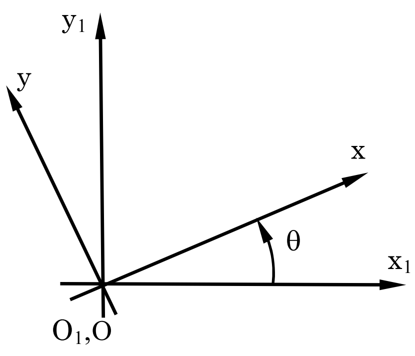


Fig.2 Rotation of rectangular coordinates

the old system $(x_1,\ y_1)$ and $c_{\ell m}$ is a direction cosine given by

$$c_{\ell m}=\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} \quad (54)$$

The complex representation related to the conductivity coefficients formulas are obtained by using Eq.(53) and (54) as follows (Appendix C):

$$\left.\begin{aligned}&K'_{11}=\kappa'_{11}-\kappa'_{22}+2i\kappa'_{12}=(\kappa_{11}-\kappa_{22}+2i\kappa_{12})e^{-2i\theta}=\\ &=K_{11}e^{-2i\theta},K'_{22}=\overline{K'_{11}}=K_{22}e^{-2i\theta}\\ &K'_{12}=\kappa'_{11}+\kappa'_{22}=\kappa_{11}+\kappa_{22}=K_{12},\ (\text{invariant})\\ &\gamma'_3=\gamma_3e^{-2i\theta}\end{aligned}\right\} \quad (55)$$

It should be noted that $K'_{12}=K_{12}$ is invariant for an arbitrary angle θ. Similarly, the transformation formulas of the thermal expansion coefficients $\alpha_{ij},\ (i,j=1,2)$ are found in similar form of Eq.(55):

$$\left.\begin{aligned}&H'_{11}=\alpha'_{11}-\alpha'_{22}+2i\alpha'_{12}=(\alpha_{11}-\alpha_{22}+2i\alpha_{12})e^{-2i\theta}=\\ &=H_{11}e^{-2i\theta},\ H'_{22}=\overline{H'_{11}}=H_{22}e^{-2i\theta}\\ &H'_{12}=\alpha'_{11}+\alpha'_{22}=\alpha_{11}+\alpha_{22}=H_{12},\ (\text{invariant})\end{aligned}\right\} \quad (56)$$

The forth-order tensor of elastic constants s_{ijkl} are transformed by

$$s'_{ijkl}=c_{im}c_{jn}c_{ko}c_{\ell p}s_{mnop} \quad (57)$$

Similar to the derivation of Eqs.(55) and (56), the following formulas for the elastic constants and the characteristic values $\gamma_1,\ \gamma_2$ are obtained,

$$\left.\begin{aligned}&(S^{11}_{11})'=S^{11}_{11},\ (S^{11}_{22})'=S^{11}_{22}e^{-4i\theta},\ (S^{11}_{12})'=S^{11}_{12}e^{-2i\theta},\\ &(S^{12}_{12})'=S^{12}_{12},\end{aligned}\right\} \quad (58)$$

$$\gamma'_\alpha=\gamma_\alpha e^{-2i\theta} \quad (59)$$

where we denote elastic constants in the new coordinates (x,y) with a prime, and without that in the old coordinates (x_1,y_1,) (Appendix D). Moreover for the constants in the basic equations of Eqs.(41)~(45), it is verified that the following expressions hold (Appendix E):

$$\left.\begin{aligned} \delta_i' &= \delta_i e^{-2i\theta} \quad (i=1,\ 2) \\ \rho_i' &= \rho_i \quad\quad (i=1,\ 2)\ \text{(invariant)} \\ \gamma_i' &= \gamma_i e^{-2i\theta} \quad (i=1,2,3) \end{aligned}\right\} \tag{60}$$

where the prime denotes the quantities related to a new system. Also, for the complex potential functions, if the new system $t(=x+iy)$ is rotated with respect to the old one $z_1(=x_1+iy_1)$ by θ, using Eqs.(55)-(60), we obtain

$$\Omega_1''(\zeta_1) = \left(\frac{1+\gamma_1}{1+\gamma_1'}\right)^2 e^{-2i\theta}\Omega''(\eta_1) \tag{61}$$

$$\omega_1''(\zeta_2) = \left(\frac{1+\gamma_2}{1+\gamma_2'}\right)^2 e^{-2i\theta}\Omega''(\eta_2) \tag{62}$$

$$\psi_1''(\zeta_3) = \left(\frac{1+\gamma_3}{1+\gamma_3'}\right)^2 e^{-2i\theta}\psi''(\eta_3) \tag{63}$$

where the complex potentials in the right hand side of Eqs.(61)-(63) are new potentials with respect to the new systems, and

$$\eta_j = \frac{t+\gamma_j'\bar{t}}{1+\gamma_j'},\ (j=1,2,3),\ t=x+iy \tag{64}$$

$$\zeta_j = \left\{\frac{t+\gamma_j'\bar{t}}{1+\gamma_j'}\right\}\eta_j e^{i\theta},\ (j=1,2,3) \tag{65}$$

By using these basic equations and transformation formulas, complex potential functions in various fracture mechanics analyses of anisotropic thermoelasticity problems will be constructed effectively in connection with the singular stress fields. For example, insulated cracks in arbitrary directions in anisotropic media under stationary uniform heat flow can be simulated by continuous distributions of edge dislocations as given in the early papers[16], and a system of singular integral equations of Cauchy type may be obtained.

Equations (50), (61), (62) and (63) correspond to the isotropic and isothermal elasticity formula obtained by Mushkelishvili[2], and the anisotropic, isothermal case of Stroh formalism was considered by Ting[17]. The above transformation formulas obtained are summarized in the Table 1 together with the isotropic, thermal complex potential functions. In addition, the case of translation and rotation occurring simultaneously, is also shown in the Table 1.

6. Fundamental solutions

6.1 Uniform heat flux q_0 at infinity with an angle β from the positive direction of the x axis.

As shown in **Fig.3**, we seek the complex potentials in a case of uniform heat flux q_0 at infinity. The Fourier's law of heat conduction in anisotropic medium can be written by using Eqs.(5) and (10) as follows:

$$\begin{aligned} F = f^1 + if^2 = f_x + if_y = q_0\cos\beta + iq_0\sin\beta = \\ = -K_{11}\frac{\partial T}{\partial z} - K_{12}\frac{\partial T}{\partial \bar{z}} = i\left[K\theta(\zeta_3) - \overline{K\theta(\zeta_3)}\right] \end{aligned} \tag{66}$$

where

$$K = \frac{1}{1+\gamma_3}\{iK_{21}(1+\gamma_3) - K_{22}(1-\gamma_3)\} \tag{67}$$

and K_{21} and K_{22} are the induced conductivity constants given by Eq.(21). Substituting Eq.(43) into Eq.(66),we obtain

$$\theta(\zeta_3) = \frac{q_0(1+\gamma_3)(K_2e^{i\beta} - K_1e^{-i\beta})}{K_1K_1 - K_2K_2}\zeta_3 + T_0 \tag{68}$$

where

$$K_1 = K_{11} + \gamma_3 K_{12},\ K_2 = K_{12} + \gamma_3\overline{K}_{11} \tag{69}$$

The function $\psi'(\zeta_2)$ will be obtained by the relation Eq.(40). In Eq.(68), the constant T_0 denotes the constant temperature rise or fall of the medium. The other two complex potentials $\Omega'(\zeta_1)$ and $\omega'(\zeta_2)$ will become zero.

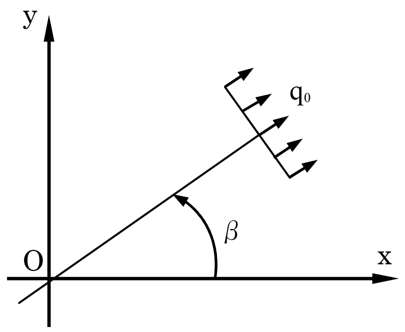


Fig.3 Uniform heat flux q_0 at infinity with angle β from the positive x axis.

6.2 A single temperature dislocation in an infinite anisotropic medium

We assume that the dislocation is located at the origin of the rectangular coordinates (x,y). The potential will be constructed by the following conditions (a) and (b):

(a) The magnitude of temperature dislocation T_0 arises when ζ_3 goes round the dislocation along an closed curve C,

$$\int_C dT = T_0 \tag{70}$$

(b) The magnitude of total heat flow is zero when ζ_3 goes round the center of temperature dislocation along an arbitrary closed curve C,

$$\int_C f_n ds = 0 \tag{71}$$

where f_n is a heat flux in the normal direction n of the curve C. The temperature potentials which satisfy the Eqs.(70) and (71) simultaneously, are given by using Eqs.(43) and (66), as follows:

$$\theta(\zeta_3) = -\frac{i T_0 \overline{K}}{2\pi(K+\overline{K})} \log \zeta_3 \tag{72}$$

The complex potentials $\Omega'(\zeta_1)$ and $\omega'(\zeta_2)$ corresponding to the temperature potential Eq.(72) can be obtained by the conditions: (c) the displacement dislocation around the temperature dislocation is zero, and (d) the total resultant force around the temperature dislocation is also zero. The result is $\Omega'(\zeta_1) = \omega(\zeta_2) = 0$.

6.3 A single concentrated heat source in a anisotropic infinite medium

We assume that the line heat source with a total quantity of heat Q per unit length is located at the origin of the rectangular coordinates (x, y). In this case the temperature potential function $\theta(\zeta_3)$ can be derived by the following conditions:

(a) The line integral of heat flux around the center of line heat source becomes the total heat flux Q.

(b) The magnitude of temperature dislocation is zero when ζ_3 goes round the line heat source along an arbitrary closed curve C.

The temperature potential function $\theta(\zeta_3)$ is obtained by using the condition (a) and (b) simultaneously, as follows:

$$\theta(\zeta_3) = -\frac{Q}{2\pi(K+\overline{K})} \log \zeta_3 \tag{73}$$

or from Eq.(40),

$$\psi''(\zeta_3) = -\frac{((\gamma_3)(1+\gamma_3)^2}{4S(\gamma_3)}\theta(\zeta_3) = \frac{(1+\gamma_3)}{2} C_3 \log \zeta_3 \tag{74}$$

where

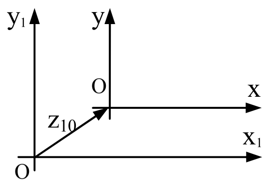
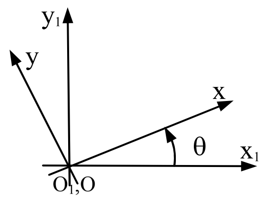
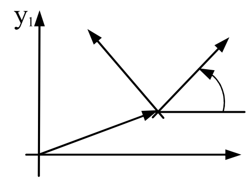
$$C_3 = \frac{Q(1+\gamma_3)H(\gamma_3)}{4\pi(K+\overline{K})S(\gamma_3)} \tag{75}$$

The complex potentials $\Omega'(\zeta_1)$ and $\omega'(\zeta_2)$ corresponding to the temperature potential $\theta(\zeta_3)$ can be obtained by the conditions: (c) the displacement dislocation around the concentrated heat is zero, and (d) the total resultant force around the concentrated heat source is zero. Under these conditions, we assume the complex potential functions in the forms with the constants C_1 and C_2:

$$\Omega'(\zeta_1) = -\frac{(1+\gamma_1)}{2} C_1 \zeta_1 \log \zeta_1 \tag{76}$$

$$\omega'(\zeta_2) = -\frac{(1+\gamma_2)}{2} C_2 \zeta_2 \log \zeta_2 \tag{77}$$

Table 1 Summary of translation and rotation formulas of complex potentials for isotropic and anisotropic thermoelasticity

	Basic equations	**Translation**	**Rotation**	**Translation and rotation**
	Stress components: $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ Displacement components: u_x, u_y Temperature :T			
Muskhelishvili potential (isotropic thermoelasticity): $\phi(z), \psi(z)$ Temperature potential: $\theta(z)$	**Stress components**: $\sigma_{xx} + \sigma_{yy} = 2\left[\phi'(z) + \overline{\phi'(z)}\right]$ $\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2\left[\bar{z}\phi''(z) + \psi'(z)\right]$ **Displacement components:** $u_x + iu_y = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} + 2\mu\beta\int\theta(z)dz$ **Temperature T:** $T(x, y) = \frac{1}{2}\left[\theta(z) + \overline{\theta(z)}\right]$	New coordinates: (x,y), t = x+iy Old coordinates:(x_1,y_1), $z=x_1+iy_1$ Translation vector: $\mathbf{z}_0$, $\mathbf{t}=\mathbf{z}+\mathbf{z}_{10}$ New complex potential: $\phi(z), \psi(z)$ New temperature potential: $\theta(z)$ Old complex potential: $\phi_1(z), \psi_1(z)$ Old temperature potential: $\theta_1(z)$ **Translation formulas**: $\phi_1'(z) = \phi(z - z_{10})$, $\psi_1'(z) = \psi_1'(z - z_{10}) - \bar{z}_{10}\phi''(z - z_{10})$ $\theta_1(z) = \theta(z - z_{10})$	**Rotation formulas:** $\phi_1'(z) = \phi(ze^{-i\theta})$, $\psi_1'(z) = \psi_1'(ze^{-i\theta})e^{-2i\theta}$ $\theta_1(z) = \theta(ze^{-i\theta})$ The signatures related to the two coordinates are the same in the case of the translation	**Translation and rotation formulas**[16]**:** $\phi_1'(z) = \phi\left\{(z - z_{10})e^{-i\theta}\right\}$, $\psi_1'(z) = \psi_1'\left\{(z - z_{10})e^{-i\theta}\right\}e^{-2i\theta} - \bar{z}_{10}\phi''\left\{(z - z_{10})e^{-i\theta}\right\}e^{-i\theta}$ $\theta_1(z) = \theta\left\{(z - z_{10})e^{-i\theta}\right\}$ The signatures related to the two coordinates are the same in the case of the translation
Anisotropic thermoelastic complex potential: (present complex potential): $\Omega(\zeta_1), \psi(\zeta_2)$ Temperature Potential: $\psi(\zeta_3)$	**Stress and strain components**: Eqs.(41),(42),(43) and (45): $\sigma_{xx} + \sigma_{yy} = \frac{4\gamma_1}{(1+\gamma_1)^2}\Omega''(\zeta_1) + \cdots + \frac{4\bar{\gamma}_3}{(1+\bar{\gamma}_3)^2}\overline{\psi(\zeta_3)}$ . . . . $u_x + iu_y = \frac{\delta_1}{1+\gamma_1}\Omega'(\zeta_1) + \cdots + \frac{\delta_3}{1+\bar{\gamma}_3}\overline{\psi(\zeta_3)}$ where $\zeta_j = \frac{z_j}{1+\gamma_j}$, $z_j = z + \gamma_j\bar{z}$ $(j=1,2,3)$	**Translation formulas**: Eqs.(50): $\Omega_1'(\zeta_1) = \Omega'(\zeta_1 - \zeta_{10})$ $\omega_1'(\zeta_2) = \omega'(\zeta_2 - \zeta_{20})$ $\psi_1'(\zeta_3) = \psi'(\zeta_3 - \zeta_{30})$ where $\zeta_j = \frac{z_j}{1+\gamma_j}$, $z_j = z + \gamma_j\bar{z}$, $(j=1,2,3)$ $\zeta_{j0} = \frac{z_{10} + \gamma_j\bar{z}_{10}}{1+\gamma_j}$, $z_{10} = x_{10} + iy_{10}$ $(j=1,2,3)$	**Rotation formulas**: Eqs.(61),(62),(63): $\Omega_1''(\zeta_1) = \left(\frac{1+\gamma_1}{1+\gamma_1'}\right)^2 e^{-2i\theta}\Omega''(\eta_1)$ $\omega_1''(\zeta_2) = \left(\frac{1+\gamma_2}{1+\gamma_2'}\right)^2 e^{-2i\theta}\Omega''(\eta_2)$ $\psi_1''(\zeta_3) = \left(\frac{1+\gamma_3}{1+\gamma_3'}\right)^2 e^{-2i\theta}\psi''(\eta_3)$ where $\eta_j = \frac{t + \gamma_j'\bar{t}}{1+\gamma_j'}$, $\zeta_j = \left\{\frac{t + \gamma_j'\bar{t}}{1+\gamma_j'}\right\}\eta_j e^{i\theta}$, t=x+iy, $(j=1,2,3)$	**Translation and rotation formulas**: $\Omega_1''(\zeta_1) = \left(\frac{1+\gamma_1}{1+\gamma_1'}\right)^2 e^{-2i\theta}\Omega''\left\{\left(\frac{1+\gamma_1}{1+\gamma_1'}\right)(\zeta_1 - \zeta_{10})e^{-i\theta}\right\}$ $\omega_1''(\zeta_2) = \left(\frac{1+\gamma_2}{1+\gamma_2'}\right)^2 e^{-2i\theta}\omega''\left\{\left(\frac{1+\gamma_2}{1+\gamma_2'}\right)(\zeta_2 - \zeta_{20})e^{-i\theta}\right\}$ $\psi_1''(\zeta_3) = \left(\frac{1+\gamma_3}{1+\gamma_3'}\right)^2 e^{-2i\theta}\psi''\left\{\left(\frac{1+\gamma_3}{1+\gamma_3'}\right)(\zeta_1 - \zeta_{30})e^{-i\theta}\right\}$ where $\zeta_j - \zeta_{j0} = \left\{\frac{1+\gamma_j'}{1+\gamma_j}\right\}\eta_j e^{i\theta}$ $(j=1,2,3)$

$$\psi'(\zeta_3) = -\frac{(1+\gamma_3)}{2} C_3 \log \zeta_3 \tag{78}$$

where C_1 and C_2 are complex constants and C_3 is given by Eq.(75). We see that the complex potentials (76), (77) and (78) will satisfy the conditions (c) and (d), if the constants C_1 and C_2 satisfy the following equations, which are derived from Eqs.(45) and (48) using above conditions.

$$\left.\begin{aligned} &\delta_1 C_1 - \rho_1 \overline{C}_1 + \delta_2 C_2 - \rho_2 \overline{C}_2 = -\delta_3 C_3 + \rho_3 \overline{C}_3 \\ &\overline{\delta}_1 \overline{C}_1 - \overline{\rho}_1 C_1 + \overline{\delta}_2 \overline{C}_2 - \overline{\rho}_2 C_2 = -\overline{\delta}_3 \overline{C}_3 + \overline{\rho}_3 C_3 \\ &\gamma_1 C_1 - \overline{C}_1 + \gamma_2 C_2 - \overline{C}_2 = \overline{C}_3 + \gamma_3 C_3 \\ &\overline{\gamma}_1 \overline{C}_1 - C_1 + \overline{\gamma}_2 \overline{C}_2 - C_2 = C_3 + \overline{\gamma}_3 \overline{C}_3 \end{aligned}\right\} \tag{79}$$

where δ_j, ρ_j (j=1, 2, 3) are given by Eqs.(46) and (47), respectively. From Eq.(79), $C_1, \overline{C}_1, C_2$ and $\overline{C}_2$ will be determined completely.

7. Conclusion

Basic analyses on the anisotropic thermo-elasticity theory have been made on the basis of the Green and Zerna's complex variable approach[1). First, based on the approach, the stress and displacement components were expressed by the three complex potentials functions. Special attention was paid to the transformation formulas of the complex potentials and the physical constants attendant on the coordinate translation and rotation of the rectangular coordinate systems. These formulas correspond to the case of Muskhelishvili's transformation formulas[2) for the isothermal, isotropic body. These will be very convenient to construct the three complex potentials in the case of using multiple rectangular coordinates for the various boundary value problems of anisotropic thermoelasticity. Finally, although well known simple solutions, the fundamental solutions were given by the complex potential obtained.

Acknowledgement : The author wishes to thank Prof. Emeritus H.Sekine of Tohoku University for his kind advice.

Appendix

A

In this case, we must replace the elastic constant s_{ij} for the plane stress with the plane strain $\tilde{s}_{ij}$ as

$$\tilde{s}_{ij} = s_{ij} - s_{i3} s_{j3} / s_{33} \tag{a1}$$

where the elastic constants s_{ij} is given by Eq.(20).

B

From Eqs.(2) and (3),

$$F^{\alpha\beta}_{\lambda\mu} \tau^{\lambda\mu} + G^{\alpha\beta} T = \frac{1}{2}\left(a^{\alpha\lambda} v^{\beta}\big|_{\lambda} + a^{\beta\lambda} v^{\alpha}\big|_{\lambda} \right) \tag{a2}$$

where $a^{\alpha\beta}$ is a metric tensor in two dimension and exchanging into the complex coordinates, $F^{\alpha\beta}_{\lambda\mu} \to S^{\alpha\beta}_{\lambda\mu}$, $v^{\alpha} \to D^{\alpha}$, $G^{\alpha\beta} \to H^{\alpha\beta}$ by Eqs.(8) ~ (9)

$$F^{\alpha\beta}_{\lambda\mu} T^{\lambda\mu} + G^{\alpha\beta} T = S^{\alpha\beta}_{\lambda\mu} T^{\lambda\mu} + H^{\alpha\beta} T = \frac{1}{2}\left(a^{\alpha\lambda} D^{\beta}\big|_{\lambda} + a^{\beta\lambda} D^{\alpha}\big|_{\lambda} \right) = \\ = \frac{1}{2}\left(a^{\alpha\lambda} D^{\beta}_{,\lambda} + a^{\beta\lambda} D^{\beta}_{,\lambda} \right) \tag{a3}$$

We put $\tau^{11} = T^{11}, \tau^{12} = T^{12}, \tau^{22} = T^{22}$ and $\alpha = = \beta = 1$ in Eq.(a2), we obtain

$$S^{11}_{11} T^{11} + S^{11}_{12} T^{12} + S^{11}_{21} T^{21} + S^{11}_{22} T^{22} + H^{11} T = \\ = \frac{1}{2}\left(a^{11} v^{1}\big|_{1} + a^{12} v^{1}\big|_{2} + a^{11} v^{1}\big|_{1} + a^{12} v^{1}\big|_{12} \right) \tag{a4}$$

where property of symmetry of strain tensor and Eqs.(8), (9), (11), (12) were used. Thus, from Eq.(a3), using Eq.(18), (19) and $a^{ii} = 0 \ (i = 1,2)$, $a^{12} = 2$, $a^{12} = 1/2$, we obtain

$$S^{11}_{11} \Phi + S^{11}_{22} \overline{\Phi} + 2 S^{11}_{12} \Theta + + H^{11} T = \\ = 2 v^{1}\big|_{2} + H^{11} T = 2 D^{1}_{,2} + H^{11} T = 2 \frac{\partial D}{\partial \overline{z}} + H^{11} T \tag{a5}$$

The other equations (25) and (26) will be derived similar procedure to Eq.(a4).

C

The new conductivity constant κ'_{ij} after rotation of the old coordinates are given by Eq.(53) as follows:

$$\kappa'_{11} = c_{1r} c_{1s} \kappa_{rs} = \\ = c_{11} c_{11} \kappa_{11} + c_{11} c_{12} \kappa_{12} + c_{12} c_{11} \kappa_{21} + c_{12} c_{12} \kappa_{22} = \\ = \cos^2 \theta \kappa_{11} + \cos\theta \sin\theta \kappa_{12} + \cos\theta \sin\theta \kappa_{21} +$$

$$+\sin^2\theta\kappa_{22} = \kappa_{11}\cos^2\theta + \kappa_{12}\sin 2\theta + \kappa_{22}\sin^2\theta \qquad \text{(a6)}$$

Similarly,

$$\kappa'_{21} = -\kappa_{11}\sin 2\theta/2 + \kappa_{12}\cos 2\theta + \kappa_{22}\sin 2\theta/2 \qquad \text{(a7)}$$

$$\kappa'_{22} = \kappa_{11}\sin^2\theta - \kappa_{12}\sin 2\theta + \kappa_{22}\cos^2\theta \qquad \text{(a8)}$$

Therefore, substituting Eqs.(a6), (a7) and (a8) into Eq.(21) corresponding the new coordinate systems, the results are

$$\left.\begin{aligned} K'_{11} &= \kappa'_{11} - \kappa'_{22} + 2i\kappa'_{12} = \kappa_{11}(\cos^2\theta - \sin^2\theta) + \\ &+ 2\kappa_{12}\sin 2\theta - \kappa_{22}(\cos^2\theta - \sin^2\theta) - i\kappa_{11}\sin 2\theta + \\ &+ 2i\cos 2\theta + i\kappa_{22}\sin 2\theta = \\ &= (\kappa_{11} - \kappa_{22} + 2i\kappa_{12})e^{-2i\theta} = K_{11}e^{-2i\theta}, \quad K'_{11} = K_{11}e^{-2i\theta} \end{aligned}\right\} \qquad \text{(a9)}$$

Similarly,

$$K'_{22} = K_{22}e^{2i\theta}, \quad K'_{12} = K_{12} \quad \text{(invariant)} \qquad \text{(a10)}$$

and

$$\gamma' = \gamma e^{-2i\theta} \qquad \text{(a11)}$$

D

Using Eq.(57), the new elastic constants s'_{ij} $(i, j = 1,2,6)$ after rotation θ of the old coordinates are given by the old elastic constants s_{ij} $(i, j = 1,2,6)$ as follows[3)] if we write only s'_{11}

$$\begin{aligned} s'_{11} &= s_{11}\cos^4\theta + s_{22}\sin^4\theta + (2s_{12} + s_{66}) \times \\ &\times \sin^2 2\theta/2 + +(s_{16}\cos^2\theta + s^{26}\sin^2\theta)\sin 2\theta \end{aligned} \qquad \text{(a12)}$$

Substituting s'_{ij} (i,j=1,2,6) into Eq.(18) and (19), we obtain after some manipulations

$$\begin{aligned} S'^{11}_{11} &= S'^{22}_{22} = S'_3 = (s'_{11} + s'_{22} + s'_{66} - 2s'_{12})/4 = \\ &= (s_{11} + s_{22} + s_{66} - 2s_{12})/4 = S^{11}_{11} \quad \text{(invariant)} \end{aligned} \qquad \text{(a13)}$$

The other expressions of Eqs.(58) and (59) will be found from the similar manner.

E

By using Eqs.(a13) and the related formulas[3)], we obtain from Eq.(46) as follows:

$$\begin{aligned} \delta'_j &= -2(S'_3\gamma'^2_j - 2S'_2\gamma'_j + S'_1)/\gamma'_j - 4H'_{11}S'(\gamma_j)/\gamma'_jH'(\gamma_j) = \\ &= -2(S_3\gamma^2_j - 2S_2\gamma_jS_1)/\gamma_j \cdot e^{-2i\theta} - 4H_{11}S(\gamma_j)e^{-2i\theta}/\gamma_jH(\gamma_j) = \\ &= \delta_je^{-2i\theta} \end{aligned} \qquad \text{(a14)}$$

Similarly, the other transformation formulas of the induced constants will be obtained.

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(Received September 30, 2015)